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# THE CONSTRUCTION OF BOUNDARY ANALOGUES OF VARIATIONAL METHODS TO APPROXIMATE WEAK SOLUTIONS OF BOUNDARY-VALUE PROBLEMS IN THE THEORY OF ELASTICITY†

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Methods of approximating weak solutions of certain boundary-value problems in the theory of elasticity are proposed based on expanding the approximate solution in a finite series in basis functions which identically satisfy a homogeneous differential equation in the domain. The coefficients of the expansion are found by constructing a boundary analogue of the method of least squares (BAMLS). It is proved that the approximate solution thus obtained converges to a weak solution of the problem. Sufficient conditions for the stability of the BAMLS, easily verifiable by computational means, are derived. The construction of a boundary analogue of the collocation method (BACM) is proposed on the basis of the BAMLS, combined with discretization of the scalar product by quadrature formulae. The BACM obtained is convergent and stable and possesses better computational properties than the BAMLS. 

## 1. STATEMENT OF THE PROBLEM

Let $\Omega$ be a bounded domain in $R^n$ ($n = 1, 2, \ldots$) with boundary $\Gamma$, let $k$ be a natural number, and let $A$ be an elliptic differential operator of order $2k$

$$A = \sum_{|i|, |j| \leq k} (-1)^{|i|} D^i(a_{ij}(x) D^j) \tag{1.1}$$

$$x = (x_1, \ldots, x_n) \in \Omega, \quad i = (i_1, \ldots, i_n), \quad j = (j_1, \ldots, j_n), \quad D^i = \frac{\partial^{|i|}}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}$$

where $\nu$ is the normal to $\Gamma$ and $g_r(s)$ are given functions on the boundary, $s \in \Gamma$; throughout, $r = 0, \ldots, k-1$. It is required to approximate the solution of the generalized Dirichlet problem for the homogeneous differential equation

$$Au(x) = 0, \quad x \in \Omega, \quad \partial^r u / \partial \nu^r |_\Gamma = g_r(s), \quad s \in \Gamma \tag{1.2}$$

We introduce additional assumptions concerning the data of problem (1.2) and consider the weak formulation of the problem. It will be assumed that $\Gamma \in \mathfrak{R}^{0,1}$ (for $k = 1$) and $\Gamma \in \mathfrak{R}^{k,1}$ (for $k > 1$), that is, the functions defining the boundary of the domain in local coordinates belong to class $C^{0,1}$ (for $k = 1$) and class $C^{k,1}$ (for $k > 1$) (see [1]), $g_r \in W_2^{k-r-1/2}(\Gamma)$, $a_{ij} \in C^\infty(\Omega)$, $|i|$, $|j| \leq k$. We introduce a bilinear form

$$A(v, u) = \int_\Omega \sum_{|i|, |j| \leq k} a_{ij}(x) D^i v(x) D^j u(x) dx \tag{1.3}$$

and a subspace $W_2^k(\Omega)$

$$V = \{v \in W_2^k(\Omega) | \partial^r v / \partial \nu^r |_\Gamma = 0\}$$

We shall assume that the form $A(v, u)$ is $V$-elliptic and bounded. The weak formulation of problem (1.2) is to determine a function $u \in W_2^k(\Omega)$ such that

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$$A(v,\ u)=0,\quad \forall v\in V,\quad u-w\in V$$
$$w\in W_2^k(\Omega),\quad \partial^r w/\partial \nu^r\,|_\Gamma = g_r(s),\quad s\in\Gamma \tag{1.4}$$

## 2. REPRESENTATION OF THE APPROXIMATE SOLUTION

We shall write an approximate solution of problem (1.4) in the form

$$u_N(x)=\sum_{j=1}^{N} a_{jN}\varphi_j(x) \tag{2.1}$$

where the functions $\varphi_j(x)$ ($i = 1, 2, \ldots$) are classical linearly independent solutions of the homogeneous differential equation $Au(x) = 0$. We shall call them global basis functions.

To abbreviate the formulae, we introduce the notation

$$\|\cdot\|_\Omega=\|\cdot\|_{W_2^k(\Omega)},\quad \|\cdot\|_r=\|\cdot\|_{W_2^{k-r-1/2}(\Gamma)}$$

Summation over $r$ is performed throughout from zero to $k - 1$.

The determination of the coefficients of expansion (2.1) reduces to approximating the boundary conditions of problem (1.2). The linearity of the boundary-value problem, in combination with previous results [2], implies the estimate

$$\|u-u_N\|_\Omega \leqslant \text{const}\sum_r\left\|g_r-\frac{\partial^r u_N}{\partial \nu^r}\right\|_r \tag{2.2}$$

We will find the coefficients of expansion (2.1) subject to the condition of the boundary analogue of the method of least squares method (BAMLS)

$$\min_{a_{1N},\ldots,a_{NN}}\ \sum_r\left\|g_r-\frac{\partial^r u_N}{\partial \nu^r}\right\|_r^2 \tag{2.3}$$

## 3. THE CONVERGENCE OF THE SOLUTION BY THE BAMLS TO THE EXACT SOLUTION

In relation to the convergence of an approximate solution (2.1), obtained using the BAMLS in the form (2.3), to a weak solution of problem (1.4), we shall assume that the form $A(\upsilon, u)$ defined by (1.3) satisfies the following condition. Let $\Omega_t$ be a sequence of bounded domains in $R^n$ with boundaries $\Gamma_t$ satisfying the conditions

$$\Gamma_t\in\mathfrak{N}^{0,1}\quad\text{for}\quad k=1,\quad \Gamma_t\in\mathfrak{N}^{k,1}\quad\text{for}\quad k>1$$
$$\overline{\Omega}\subset\Omega_t,\quad \overline{\Omega}_{t+1}\subset\Omega_t,\quad \lim_{t\to\infty}\operatorname{mes}(\Omega_t/\Omega)=0 \tag{3.1}$$

Throughout, $t = 1, 2, \ldots$. We shall assume that the coefficients $a_{ij}(x)$ of the form $A(\upsilon, u)$ satisfy the conditions $a_{ij}(x) \in C^\infty(\Omega)$, $|i|$, $|j| \leqslant k$. We introduce notation

$$V_t=\left\{v\in W_2^k(\Omega_t)\left|\frac{\partial^r v}{\partial \nu^r}\right|_{\Gamma_t}=0\right\},\quad A_t(v,\ u)=\int_{\Omega_t}\sum_{|i|,|j|\leqslant k} a_{ij}(x)D^i v(x)D^j u(x)dx$$

Let the forms $A_t(\upsilon, u)$ be $V$-elliptic and bounded

$$A_t(v,\ v)\geqslant\alpha_t\|v\|_{\Omega_t}^2,\quad |A_t(v,\ u)|\leqslant\beta_t\|v\|_{\Omega_t}\|u\|_{\Omega_t} \tag{3.2}$$

where positive constants $\alpha$ and $\beta$ exist such that $\alpha < \alpha_t$ and $\beta > \beta_t$.

The following auxiliary propositions generalize certain propositions established in [3] for $A = \Delta$ (where $\Delta$ is the Laplacian) and $k = 2$ and $n = 2$.

*Lemma* 1. Let $u \in W_2^k(\Omega)$ be a weak solution of problem (1.4) and suppose a sequence of domains $\Omega_t$ satisfying conditions (3.1) and (3.2) exists. Then, for any $\varepsilon > 0$ a bounded domain $\Omega'$ in $R^n$, and a function $u_0 \in C^\infty(\Omega')$ exist such that

$$\overline{\Omega} \subset \Omega', \quad Au_0(x) \equiv 0, \quad x \in \Omega', \quad \|u - u_0\|_{\Omega'} < \varepsilon$$

*Lemma* 2. For any function $z \in W_2^k(\Omega)$ and any positive number $\varepsilon$ a domain $\Omega'$ and a function $\tilde{z} \in C^\infty(\Omega')$ exist such that $\Omega^- \subset \Omega'$, $A\tilde{z}(x) \equiv 0, x \in \Omega'$, and

$$\left\| \frac{\partial^r \tilde{z}}{\partial v^r} - \frac{\partial^r z}{\partial v^r} \right\|_r < \varepsilon$$

Thus, the problem of the convergence of the BAMLS reduces to the problem of approximating the classical solution of an elliptic differential equation.

We will consider an example which illustrates the construction of a system of global basis functions leading to a convergent BAMLS.

*The Dirichlet problem for a polyharmonic equation.* Various problems of the theory of elasticity can be reduced to a Dirichlet problem for harmonic or biharmonic equations: the twisting of a prism, the buckling of a plate (with prescribed deflection and bending angle at the boundary), a plane stressed state and plane deformation. We will consider the general case: approximating the solution of the generalized Dirichlet problem for a polyharmonic equation.

Let $\Delta$ be the $n$-dimensional Laplacian. We will consider problem (1.2) for $A \equiv \Delta^k$. Note that the coefficients $a_{ij}(x)$ of the form $A(\upsilon, u)$ corresponding to the polyharmonic operator $\Delta^k$ are constant and can be continued into $R^n/\Omega$. We now verify that the sequence of domains (3.1) and the form $A(\upsilon, u)$ satisfy conditions (3.2). Without loss of generality, we will confine our attention to the case $n = 2$, $k = 2$. In that case

$$A_t(\upsilon, u) = \int\limits_{\Omega_t} \left[ \frac{\partial^2 \upsilon}{\partial x_1^2} \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 \upsilon}{\partial x_1 \partial x_2} \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 \upsilon}{\partial x_2^2} \frac{\partial^2 u}{\partial x_2^2} \right] dx$$

and the fact that the form $A_t(u, \upsilon)$ is $W_2^{2,0}(\Omega_t)$-elliptic is established using the Friedrichs inequality. Let $d_t = \operatorname{diam} \Omega_t$. If $\upsilon \in W_2^{2,0}(\Omega_t)$, we have an estimate

$$\|\upsilon\|_{\Omega_t}^2 \leq \alpha_t^{-1} |\upsilon|_{\Omega_t}^2, \quad \alpha_t = [d_t^4/4 + d_t^2/2 + 1]^{-1}$$

where $|\cdot|_{\Omega_t}$ is the norm defined by the highest-order derivative. Then, since, $|A_t(\upsilon, \upsilon)| = |\upsilon|_{\Omega_t}^2$, it follows that

$$|A_t(\upsilon, \upsilon)| \geq \alpha_t \|\upsilon\|_{W_2^k(\Omega_t)}^2 \tag{3.3}$$

It follows from (3.1) that $0 < d < d_{t+1} < d$ and a positive number $\alpha$ exists such that $\alpha_t > \alpha$, and the first condition of (3.2) is satisfied. The second condition also holds.

We will now consider the convergence of the BAMLS. Let $u(x)$ denote a weak solution of problem (1.2) for $A \equiv \Delta^k$. Choose any positive number $\varepsilon$. By Lemma 1, a bounded domain $\Omega'$ in $R^n$ and a function $\tilde{u} \in C^\infty(\Omega')$ exist such that

$$\overline{\Omega} \subset \Omega', \quad \Delta^k \tilde{u}(x) \equiv 0, \quad x \in \Omega', \quad \|u - \tilde{u}\|_{\Omega'} < \varepsilon \tag{3.4}$$

The function $\tilde{u}(x)$, which is polyharmonic in $\Omega'$, may be expressed by Almansi's formula [4] as follows (throughout, summation over $m$ is performed from zero to $k - 1$)

$$\tilde{u}(x) = \sum_m |x|^{2m} \tilde{u}_m(x), \quad |x|^{2m} = \left[ \sum_{l=1}^n x_l^2 \right]^m$$

where $\tilde{u}_m(x)$ are functions harmonic in $\Omega'$. Let $\{P_q(x)\}_{q=1,2,\ldots}$ denote the system of harmonic polynomials constructed in [5]. It was proved there that any harmonic function may be uniformly approximated to any degree of accuracy by a certain linear combination of elements of the system $\{P_q(x)\}_{q=1,2,\ldots}$. Let

$$\tilde{\Sigma}_{mM}=\sum_{q=1}^{M}\tilde{a}_{qm}P_q(x)$$

Let $\tilde{\Sigma}_{mM}\to\tilde{u}_m(x)$ as $M\to\infty$ uniformly in $\Omega'$, $m=0,\ldots,k-1$. Then [6] for any domain $\Omega$ such that $\bar{\Omega}\subset\Omega'$, $D^i\tilde{\Sigma}_{mM}\to D^i\tilde{u}_m(x)$ as $M\to\infty$ uniformly in $\Omega'$, $m=0,\ldots,k-1$, $|i|=0,1,\ldots$.

For the selected value of $\varepsilon$, an $N=kM$ exists such that

$$\|\tilde{u}_N-\tilde{u}\|_\Omega<\varepsilon,\quad \tilde{u}_N(x)=\sum_m|x|^{2m}\sum_{q=1}^{M}\tilde{a}_{qm}P_q(x) \tag{3.5}$$

It follows from the triangle inequality, (3.4) and (3.5) that $\|\tilde{u}_N-u\|_\Omega<2\varepsilon$. The Embedding Theorem for manifolds [11] implies the limit

$$\left\|\frac{\partial^r\tilde{u}_N}{\partial\nu^r}-\frac{\partial^r u}{\partial\nu^r}\right\|_r<\mathrm{const}\cdot\varepsilon \tag{3.6}$$

Taking into account that the equality $\partial^r u/\partial\nu^r=g_r$ holds on $\Gamma$, we deduce from (3.6) that

$$\sum_r\left\|\frac{\partial^r\tilde{u}_N}{\partial\nu^r}-g_r\right\|_r^2<\mathrm{const}\cdot\varepsilon^2$$

We can strengthen this inequality by replacing $\tilde{u}_N(x)$ by

$$u_N(x)=\sum_m|x|^{2m}\Sigma_{mM},\quad \Sigma_{mM}=\sum_{q=1}^{M}a_{qmN}P_q(x) \tag{3.7}$$

here $N=kM$ and the coefficients $a_{qmN}$ are determined from the condition of the BAMLS (2.3), which in this case has the form

$$a_{qmN},\quad q=1,\ldots,\min_M,\quad m=0,\ldots,k-1\sum_r\left\|\frac{\partial^r u_N}{\partial\nu^r}-g_r\right\|_r^2$$

Then we obtain the estimate

$$\sum_r\left\|\frac{\partial^r u_N}{\partial\nu^r}-g_r\right\|_r^2<\mathrm{const}\cdot\varepsilon^2 \tag{3.8}$$

We write expression (3.7) in the form

$$u_N(x)=\sum_m\sum_{q=1}^{M}a_{qmN}|x|^{2m}P_q(x) \tag{3.9}$$

Let us renumber the terms on the right of (3.9). Let $j=j(m,q)$ ($j=1,\ldots,N$) and suppose

$$a_{jN}=a_{qmN},\quad \varphi_j(x)=|x|^{2m}P_q(x)$$

Then formula (3.9) takes the form of the expansion of the approximate solution in a series in the global basis functions (2.1). It follows from estimates (2.2) and (3.8) that

$$\|u_N-u\|_\Omega\to 0\quad\text{as}\quad N\to\infty$$

This shows that the approximate solution given by the BAMLS for the weak formulation of the generalized Dirichlet problem for a polyharmonic equation indeed converges to the exact solution.

We will proceed as follows: We will consider the numerical implementation of the BAMLS, prove that the system of linear algebraic equations of the BAMLS is solvable and derive computationally simple conditions for the global basis functions which are sufficient for the BAMLS to be computationally stable. Then, on the basis of the convergent and stable boundary analogue of the BAMLS, we will construct a convergent and stable boundary analogue of the collocation method (BACM), to approximate weak solutions of problem (1.4), with better computational properties than the BAMLS.

## 4. NUMERICAL IMPLEMENTATION OF THE BAMLS

Conditions (2.3) lead to a system of linear algebraic equations

$$\mathbf{M}^{(N)}\mathbf{a}^{(N)} = \mathbf{g}^{(N)} \quad (p,\ q = 1,\ \ldots,\ N) \tag{4.1}$$

where the matrix $\mathbf{M}^{(N)}$ and the vector $\mathbf{g}^{(N)}$ have components

$$M_{pq}^{(N)} = \sum_r \left(\frac{\partial^r \varphi_q}{\partial \nu^r},\ \frac{\partial^r \varphi_p}{\partial \nu^r}\right)_r,\quad g_p^{(N)} = \sum_r \left(g_r,\ \frac{\partial^r \varphi_p}{\partial \nu^r}\right)_r;\quad \mathbf{a}^{(N)} = [a_{1N},\ \ldots,\ a_{NN}]^T$$

*Theorem* 1. System (4.1) is uniquely solvable for any natural number $N$.

The proof of this theorem is analogous to the proof that the system of the BAMLS is solvable in the case of the approximation of a very weak solution of a biharmonic problem [2, 3]. It can be shown that $\mathbf{M}^{(N)}$ is the Gram matrix of a system of global basis functions in a Hilbert space with scalar product defined by

$$\sum_r \left(\frac{\partial^r u}{\partial \nu^r},\ \frac{\partial^r v}{\partial \nu^r}\right)_r = (u,\ v)_\Gamma$$

## 5. NUMERICAL STABILITY OF THE BAMLS

In the numerical implementation, the elements of the matrix $\mathbf{M}^{(N)}$ and the vector $\mathbf{g}^{(N)}$ on the right-hand side of system (4.1) are determined to within a certain error, which arises when the definite integrals are computed using quadrature formulae (when $n = 2$) or cubature formulae ($n > 2$). In practice, therefore, instead of system (4.1) one solves a system with a "perturbed" matrix and right-hand-side vector

$$(\mathbf{M}^{(N)} + \delta\mathbf{M}^{(N)})\mathbf{b}^{(N)} = \mathbf{g}^{(N)} + \delta\mathbf{g}^{(N)}$$

The solution $u_N(x)$ obtained using the BAMLS is replaced by the solution

$$\tilde{u}_N(x) = \sum_{j=1}^{N} b_{jN}\varphi_j(x)$$

Let us investigate the stability of the BAMLS in the sense of the definitions of [7]. Let $\|\cdot\|$ denote the Eucliden norm in $R^N$.

*Definition* 1. The BAMLS is said to be stable if constants $c_1$, $c_2$, $c_3$, independent of $N$, such that, for $\|\delta\mathbf{M}^{(N)}\| \leqslant c_1$ and any $\delta g^{(N)}$, the following inequality holds

$$\|u_N - \tilde{u}_N\|_\Gamma \leqslant c_2\|\delta\mathbf{M}^{(N)}\| + c_3\|\delta\mathbf{g}^{(N)}\|$$

The stability of the BAMLS depends on the choice of the system of global basis functions.

*Definition* 2. A system of functions $\{\varphi_j(x)\}_{j=1,2,\ldots}$ is said to be strongly minimal in a Hilbert space if a positive constant $\lambda_0$ such that $\lambda_{\min, N} \geqslant \lambda_0$ for all natural $N$, where $\lambda_{m,N}$ is the minimum eigenvalue of the Gram matrix of the first $N$ elements of the system $\{\varphi_j(x)\}_{j=1,2,\ldots}$ in the Hilbert space.

The following theorem holds [7].

*Theorem* 3. Let the domain $\omega \subset R^n$ be such that $\gamma \in \mathfrak{R}^{l,1}$, $\omega^- \subset \Omega$, $l$ being a non-negative integer with $l < k$, and let the system of global basis functions $\{\varphi_j(x)\}_{j=1,2,\ldots}$ be strongly minimal in $W_2^l(\gamma)$. Then the BAMLS in the form (2.3) is stable.

*Proof.* The minimum eigenvalue of the Gram matrix of the first $N$ functions in the system $\{\varphi_j(x)\}_{j=1,2,\ldots}$ is obtained using the variational principle for describing eigenvalues of Hermitian matrices [8]

$$\lambda_{\min,N} = \min_{\beta\neq 0}\left[\sum_{p,q} (\varphi_p, \varphi_q)_\Gamma \alpha_p \alpha_q\right]\beta^{-1}, \quad \beta = \left[\sum_p |\alpha_p|^2\right] \tag{5.1}$$

Throughout, summation with respect to both $p$ and $q$ is carried out from 1 to $N$.

We note that

$$\sum_{p,q} (\varphi_p, \varphi_q)_\Gamma \alpha_p \alpha_q = \|z_N\|_\Gamma^2, \quad z_N(x) = \sum_p \alpha_p \varphi_p(x) \tag{5.2}$$

Obviously, the function $z_N(x)$ is a solution of the boundary-value problem

$$Au(x) = 0, \quad x \in \Omega, \quad \left.\frac{\partial^r u}{\partial \nu^r}\right|_\Gamma = \left.\frac{\partial^r z_N}{\partial \nu^r}\right|_\Gamma \tag{5.3}$$

Then it satisfies an estimate of type (2.2)

$$\|z_N\|_\Omega \leqslant \text{const}\cdot \sum_r \left\|\frac{\partial^r z_N}{\partial \nu^r}\right\|_r \tag{5.4}$$

It follows from the Embedding Theorem for manifolds that

$$\|z_N\|_\Omega \geqslant \text{const}\cdot \|z_N\|_{W_2^l(\gamma)}$$

Estimate (5.4) yields

$$\|z_N\|_{W_2^l(\gamma)} \leqslant \text{const}\cdot \sum_r \left\|\frac{\partial^r z_N}{\partial \nu^r}\right\|_r$$

Squaring both sides of this inequality and taking into account that

$$\left[\sum_r c_r\right]^2 = \sum_{r,m} c_r c_m \leqslant \frac{1}{2}\sum_{r,m} (c_r^2 + c_m^2) = k\sum_r c_r^2$$

we obtain

$$\|z_N\|_{W_2^l(\gamma)}^2 \leqslant \text{const}\cdot \sum_r \left\|\frac{\partial^r z_N}{\partial \nu^r}\right\|_r^2 \tag{5.5}$$

We replace the function $z_N(x)$ in (5.5) by its expression in terms of the global basis functions (5.3) and bear in mind the definition of the scalar product $(.,.)_\Gamma$. This gives

$$\|z_N\|_{W_2^l(\gamma)}^2 \leqslant \text{const}\cdot \|z_N\|_\Gamma^2$$

This inequality holds for any values of $\alpha_1, \ldots, \alpha_N$. Dividing both sides by $\beta$ and taking the minimum of both sides over all $\beta \neq 0$, we have

$$\min_{\beta\neq 0}\left\|\sum_p \alpha_p \varphi_p\right\|_{W_2^l(\gamma)}^2 \beta^{-1} \leqslant c \min_{\beta\neq 0}\left\|\sum_p \alpha_p \varphi_p\right\|_\Gamma^2 \beta^{-1} \tag{5.6}$$

where $C$ is some positive constant. By the assumption of the theorem, the system of global basis functions

$\{\varphi_j(x)\}_{j=1,2,\ldots}$ is strongly minimal in $W_2(\gamma)$. It then follows from (5.1), (5.2) that a positive number $\mu_0$ exists such that, for all natural $N$

$$\mu_0 \le \min_{\beta\ne 0}\left\|\sum_p \alpha_p\varphi_p\right\|^2_{W_2^\prime(\gamma)} \beta^{-1}$$

But then we deduce from (5.6) that for all natural $N$

$$\lambda_0 \le \min_{\beta\ne 0}\left\|\sum_p \alpha_p\varphi_p\right\|^2_{\Gamma} \beta^{-1} = \lambda_{\min N}$$

that is, the system $\{\varphi_j(x)\}_{j=1,2,\ldots}$ is strongly minimal in a Hilbert space with scalar product $(.,.)_\Gamma$. Using Theorem 2, we conclude that the BAMLS in the form (2.3) is stable. The theorem is proved.

*Remark* 1. The theorem just proved yields simple sufficient conditions for the stability of the BAMLS. For example, it is sufficient to orthonormalize the system of global basis functions in $W_2(\gamma)$. The choice of a domain $\omega$ with a sufficiently simple boundary $\gamma$ facilitates the orthonormalization procedure.

*Remark* 2. It would have been possible to orthonormalize the system of global basis functions from the start in the Hilbert space with scalar product $( \ . \ , \ . \ )_\Gamma$. However, this approach has several drawbacks. First, the orthonormalization procedure may prove to be numerical unstable. Second, this approach turns out not to be effective in solving problems with a variable boundary. It would thus be necessary to re-orthonormalize the system of global basis functions for every change in the boundary $\Gamma$. But if one chooses a certain domain $\omega$ containing the varying domain $\Omega$ in its interior, and carries out the orthonormalization in $W_2^s(\gamma)$, then the BAMLS will be stable for any variation of the boundary $\Gamma$.

## 6. CONSTRUCTION OF A SYSTEM OF LINEAR ALGEBRAIC EQUATIONS FOR THE BAMLS

We will write the system of linear algebraic equations of the BAMLS (4.1) as follows:

$$\sum_q\left\{\sum_r\left(\frac{\partial^r\varphi_q}{\partial\nu^r},\frac{\partial^r\varphi_p}{\partial\nu^r}\right)_r\right\}a_q^{(N)} = \sum_r\left(g_r,\frac{\partial^r\varphi_p}{\partial\nu^r}\right)_r, \quad p=1,\ldots,N$$

Suppose that the sequence of domains $\{Q_j\}_{j=1,\ldots,J}$ forms a cover of the boundary $\Gamma$: $\Gamma\subset Q_1 \leqslant\ldots\leqslant Q_J$, and that on the cover $\{Q_j\}_{j=1,\ldots,J}$ we are given a partition of unity $\{\eta_j\}_{j=1,\ldots,J}$, $\eta_j\in C_0^\infty$ $(Q_j\geqslant\Gamma)$, $\eta+\ldots\eta_J\equiv 1$ on $\Gamma$. We introduce the notation

$$Q=\left\{y\in R^n \mid y=(y',y_n),\ \|y'\|<1,\ |y_n|<1\right\}$$

$$Q^-=\left\{y\in R^n \mid y=(y',y_n),\ \|y'\|<1,\ -1<y_n<0\right\}$$

$$Q^+=\left\{y\in R^n \mid y=(y',y_n),\ \|y'\|<1,\ 0<y_n<1\right\},\ Q_j^+=\Omega\cap Q_j,\ Q_j^-=\left(R^n/\Omega\right)\cap Q_j$$

Let $T_j$ be a mapping such that $T_j$: $Q^+\to Q_j^+$, $T_j$: $Q^-\to Q_j^-$ (throughout, $j=1,\ldots,J$; summation over $j$ is carried out from 1 to $J$). Let us consider the computation of the matrix elements and the vector of the right-hand sides of the system of the BAMLS.

The scalar product in the Hilbert space $W_2^s(\Gamma)$ for real $s$ is defined as [9]

$$(u,v)_{W_2^s(\Gamma)}=\sum_j\left(\eta_j(y')u(y'),\eta_j(y')v(y')\right)_{W_2^s(R^{n-1})}$$

$$u_j(y')=u\left(T_j(y',0)\right),\ v_j(y')=v\left(T_j(y',0)\right),\ \eta_j(y')=\eta_j\left(T_j(y',0)\right)$$

Allowance is made in $W_2^s(R^{n-1})$ for the definition of the scalar product using Fourier transforms. We obtain

$$(u,v)_{W_2^s(\Gamma)} = \sum_j \int_{R^{n-y}} \left(1+|\xi|^2\right)^s \widehat{\zeta_j u_j}(\xi)\overline{\widehat{\zeta_j u_j}(\xi)}d\xi$$

$$\widehat{\zeta_j w_j}(\xi) = \int_{\operatorname{supp}\zeta_j = B(0,1)} e^{-i\xi y'}\zeta_j(y')w_j(y')dy',\quad w_j = u_j, v_j$$

$$\xi y' = \xi_1 y_1 + \ldots + \xi_{n-1}y_{n-1}$$

(the bar denotes complex conjugation). Written out in full, the system of linear algebraic equations of the BAMLS has the form

$$\sum_q \Big(\sum_r \sum_j \int_{R^{n-1}} \left(1+|\xi|^2\right)^{k-r-1/2} \widehat{\zeta_j\varphi_{qrj}}(\xi)\overline{\widehat{\zeta_j\varphi_{prj}}(\xi)}\, d\xi\Big)a_q^{(N)} =$$

$$= \sum_r \sum_j \int_{R^{n-1}} \left(1+|\xi|^2\right)^{k-r-1/2} \widehat{\zeta_j g_{rj}}(\xi)\overline{\widehat{\zeta_j\varphi_{prj}}(\xi)}\, d\xi \tag{6.1}$$

$$\varphi_{prj}(y') = \partial^r\varphi_p\left(T_j(y',0)\right)/\partial v^r,\quad g_{rj}(y') = g_r\left(T_j(y',0)\right),\quad p = 1,\ldots,N$$

Note that the construction of the system of the BAMLS is not simple. On the basis of a convergent and stable BAMLS, we will now derive a convergent and stable BACM which will have better computational properties.

## 7. CONSTRUCTION OF A BACM TO APPROXIMATE THE SOLUTION OF THE DIRICHLET PROBLEM

We will consider two cases of practical importance: when the Fourier transforms of the global basis functions and the boundary conditions have already been found (and the BACM will be constructed for the Fourier transforms), and when the Fourier transforms of the global basis functions and the boundary conditions are computed using cubature formulae (and the BACM will be constructed for the basis functions themselves).

7.1. *Construction of a BACM for Fourier transforms.* Let us evaluate the integrals occurring in system (6.1) of the BAMLS using cubature formulae of order $L_r$ with coefficients $A_{lr}$ and mesh points $\xi_{lr}$ ($l = 1, \ldots, L_r$). We define a subscript $t = t(r, l)$, with $t = l, \ldots, M$, where $M = L_0 + \ldots + L_{k-1}$, and put

$$\Psi_{qjt} = S_{lr}\widehat{\zeta_j\varphi_{qrj}}(\xi_{lr}),\quad \chi_{jt} = S_{lr}\widehat{\zeta_j g_{rj}}(\xi_{lr}) \tag{7.1}$$

$$S_{lr} = \left[A_{lr}\left(1+|\xi_{lr}|^2\right)^{k-r-1/2}\right]^{1/2},\quad \sum_r\sum_j R_{pqrj} = R_{pq},\quad \sum_r\sum_j r_{prj} = r_p$$

where $R_{pqrj}$, $r_{prj}$ are the errors of the cubature formulae. Then the system of the BAMLS has the form

$$\sum_q \left\{\sum_j \sum_{t=1}^M \Psi_{qjt}\overline{\Psi}_{pjt} + R_{pq}\right\}a_q^{(N)} = \sum_j \sum_{t=1}^M \overline{\Psi}_{pjm}\chi_{jm} + r_p \tag{7.2}$$

Consider the matrices and vectors

$$\mathbf{K}_j^{(N)} = \left\{\Psi_{qjt}\right\}_{t=1,\ldots,M,q=1,\ldots N},\quad \mathbf{R}^{(N)} = \left\{\mathbf{R}_{pq}\right\}_{p,q=1,\ldots,N},\quad \mathbf{h}_j^{(N)} = \left\{\chi_{jt}\right\}_{t=1,\ldots,M},\quad \mathbf{r}^{(N)} = \left\{r_p\right\}_{p=1,\ldots,N}.$$

We express (7.2) as

$$\sum_q \left\{\sum_j \sum_{t=1}^M \left(\mathbf{K}_j^{(N)}\right)^*_{pt}\left(\mathbf{K}_j^{(N)}\right)_{tq} + \left(\mathbf{R}^{(N)}\right)_{pq}\right\}a_q^{(N)} = \sum_j \sum_{t=1}^M \left(\mathbf{K}_j^{(N)}\right)^*_{pt}\left(\mathbf{h}_j^{(N)}\right)_t + \left(\mathbf{r}^{(N)}\right)_p$$

where the matrix $\mathbf{M}^*$ is the complex conjugate of $\mathbf{M}$. Define a block matrix $\mathbf{K}^{(N)}$ and block vector $\mathbf{h}^{(N)}$ by the formulae

$$\mathbf{K}^{(N)} = \begin{bmatrix} \mathbf{K}_1^N \\ \hline \vdots \\ \hline \mathbf{K}_J^N \end{bmatrix}, \quad \mathbf{h}^{(N)} = \begin{bmatrix} \mathbf{h}_1^{(N)} \\ \hline \vdots \\ \hline \mathbf{h}_J^{(N)} \end{bmatrix}$$

Then system (7.2) becomes

$$[(\mathbf{K}^{(N)*}\mathbf{K}^{(N)}+\mathbf{R}^{(N)}]\mathbf{a}^{(N)} = (\mathbf{K}^{(N)})^*\mathbf{h}^{(N)}+\mathbf{r}^{(N)} \tag{7.3}$$

Note that systems (7.3) and (4.1) are equivalent, i.e. the matrix $\mathbf{M}^{(N)}$ and the right-hand-side vector $\mathbf{g}^{(N)}$ of system (4.1) of the BAMLS admit of the following representations

$$\mathbf{M}^{(N)} = (\mathbf{K}^{(N)})^*\mathbf{K}^{(N)} + \mathbf{R}^{(N)}, \; \mathbf{g}^{(N)} = (\mathbf{K}^{(N)})^*\mathbf{h}^{(N)} + \mathbf{r}^{(N)}$$

Together with system (7.3), we consider the system without the matrix $\mathbf{R}^{(N)}$ and the vector $\mathbf{r}^{(N)}$, containing the computation errors of the definite integrals

$$(\mathbf{K}^{(N)})^*\mathbf{K}^{(N)}\mathbf{b}^{(N)} = (\mathbf{K}^{(N)})^*\mathbf{h}^{(N)} \tag{7.4}$$

We choose the number of mesh points in the cubature formulae so as to satisfy the equality $JM = N$, that is, so that $\mathbf{K}^{(N)}$ is a square matrix. Then the solution $\mathbf{b}^{(N)}$ is identical with the solution of the system

$$\mathbf{K}^{(N)}\mathbf{b}^{(N)} = \mathbf{h}^{(N)} \tag{7.5}$$

System (7.5) is the system of a BACM with mesh points $\xi_{lr}$ for the Fourier transforms of the basis functions and boundary conditions.

We will now consider the convergence and stability of the BACM just constructed. Note that the BACM yields a sequence of approximate solutions $\tilde{u}_N(x) = b_{1N}\varphi_1(x) + \ldots + b_{NN}\varphi_N(x)$ which is generally different from the sequence $u_N(x)$ of approximate solutions obtained by using the BAMLS. We will derive the sufficient conditions for convergence of the BACM. We have to prove that $||\tilde{u}_N - u_N||_\Gamma \to 0$, as $N \to \infty$. Then, since $||u_N - u||_\Gamma \to 0$ as $N \to \infty$, where $u(x)$ is the exact solution, it will follow that the approximate solutions produced by the BACM convergence to the exact solution. System (7.4), whose solution is identical with that of the system of the BACM (7.5), may be regarded as the system of a BAMLS with a "perturbed" matrix and right-hand-side vector. The matrix and vector of the "perturbations" are $-\mathbf{R}^{(N)}$ and $-\mathbf{r}^{(N)}$, respectively. The results of Section 5 concerning the stability of the BAMLS imply the following.

*Theorem* 4. Let the cubature formulae (7.1) be such that $||\mathbf{R}^{(N)}|| \to 0$, $||\mathbf{r}^{(N)}|| \to 0$ as $N \to \infty$, and assume that the system of global basis functions $\{\varphi_j(x)\}_{j=1,2,\ldots}$ is strongly minimal in a Hilbert space with scalar product $(\cdot,\cdot)_\Gamma$. Then

$$\|\tilde{u}_N - u_N\|_\Gamma \to 0 \quad \text{as} \quad N \to \infty$$

This establishes the convergence of the BACM. We can now establish the advantages of using the BACM instead of the BAMLS. Let $\mu(\mathbf{M})$ denote the number of singularities of the matrix $\mathbf{M}$ relative to $||\cdot||$. Using Weyl's theorem [8] on the perturbation of the eigenvalues of Hermitian matrices, we can prove the following.

*Theorem* 5. Let the cubature formulae used to compute the definite integrals in system (6.1) be such that $||\mathbf{R}^{(N)}|| \to 0$, $||\mathbf{r}^{(N)}|| \to 0$ as $N \to \infty$. Then

$$\lim\mu(\mathbf{K}^{(N)}) = \lim\sqrt{\mu(\mathbf{M}^{(N)})} \quad \text{as} \quad N \to \infty$$

*7.2. Construction of a BACM for the global basis functions*. We will now consider the possibility of constructing a BACM to compute a system of linear algebraic equations with a matrix and right-hand-side vector containing (unlike those in Subsection 7) not the Fourier transforms $\zeta_j(y')\varphi_{qrj}(y')$, $\zeta_j(y')$, $g_{rj}(y')$ of the functions, but the functions themselves, evaluated at points corresponding to certain points of the boundary $\Gamma$.

We have

$$\widehat{\zeta_j\varphi_{prj}}(\xi_{lr}) = \int_{B(0,1)} e^{-i\xi_{lr}y'}\zeta_j(y')\varphi_{prj}(y')dy', \quad \widehat{\zeta_j g_{prj}}(\xi_{lr}) = \int_{B(0,1)} e^{-i\xi_{lr}y'}\zeta_j(y')g_{rj}(y')dy' \tag{7.6}$$

We will evaluate the integrals in (7.6) using cubature formulae with $L$ mesh points $y_1, \ldots, y_L$ and coefficients $B_1, \ldots, B_L$. Put

$$\Phi_{qrjlt} = S_{lr}B_t e^{-i\xi_{lr}y_t'}\zeta_j(y_t')\varphi_{qrj}(y_t')$$

$$\chi_{rjlt} = S_{lr}B_t e^{-i\xi_{lr}y_t'}\zeta_j(y_t')g_{rj}(y_t')$$

Then system (6.1) of the BAMLS has the form

$$\sum_q\sum_r\sum_j\left\{\sum_{l=1}^{L_r}\left(\sum_{t=1}^{L}\overline{\Phi}_{qrjlt}\sum_{t=1}^{L}\Phi_{qrjt}\right)+\tilde{R}_{pqrj}\right\}a_q^{(N)} = \sum_r\sum_j\left\{\sum_{l=1}^{L_r}\left(\sum_{t=1}^{L}\overline{\Phi}_{prjlt}\sum_{t=1}^{L}\chi_{rjlt}\right)+\tilde{r}_{qrj}\right\}$$

where $\tilde{R}_{pqrj}$ and $\tilde{r}_{prj}$ are the total errors in computing the definite integrals.

We now proceed as in the derivation of system (7.4) for the BACM, omitting the errors of the quadrature formulae and considering the corresponding system In that system, we invert the order of summation over $l = 1, \ldots, L_r$ and $t_1, t_2 = 1, \ldots, L$, introducing the following matrices and vectors

$$\mathbf{F}_{qrj} = \{\Phi_{qrjlt}\}_{l=1,\ldots,Lr,t=1,\ldots,L}, \quad \mathbf{H}_{rj} = \{\chi_{rjlt}\}_{l=1,\ldots,Lr,t=1,\ldots,L}$$

$$\mathbf{f}_{qrj} = \mathbf{F}_{qrj}\mathbf{e}_L, \quad \mathbf{h}_{rj} = \mathbf{H}_{rj}\mathbf{e}_L; \quad \mathbf{e}_L = [1,\ldots,1] \in R^L$$

We obtain

$$\sum_q\sum_r\sum_j \mathbf{f}_{prj}^*\mathbf{f}_{qrj}b_q^{(N)} = \sum_r\sum_j \mathbf{f}_{prj}^*\mathbf{h}_{qrj} \tag{7.7}$$

We now renumber the vectors $\mathbf{f}_{qrj}$, $\mathbf{h}_{rj}$ as $\mathbf{f}_{qm}$, $\mathbf{h}_m$ using an index transformation $t = t(r, j)$, where $t = 1, \ldots, M$, with $M = kJ$. Transforming further, taking the renumbering into account, we obtain

$$\sum_q\sum_{t=1}^{M}\mathbf{f}_{pt}^*\mathbf{f}_{qt}b_q^{(N)} = \sum_{t=1}^{M}\mathbf{f}_{pt}^*\mathbf{h}_t \tag{7.8}$$

Put

$$\mathbf{f}_q = \begin{bmatrix}\mathbf{f}_{q1}\\ \hline \vdots \\ \hline \mathbf{f}_{qM}\end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix}\mathbf{h}_1\\ \hline \vdots \\ \hline \mathbf{h}_M\end{bmatrix}, \quad \mathbf{F} = [\mathbf{f}_1 \mid \ldots \mid \mathbf{f}_N]$$

Then system (7.8), in matrix notation, is

$$\mathbf{F}^*\mathbf{F}\mathbf{b}^{(N)} = \mathbf{F}^*\mathbf{h} \tag{7.9}$$

Choose the number of mesh points in the cubature formulae so that $J(L_0 + \ldots + L_{k-1}) = N$. Then the matrices $\mathbf{F}$ and $\mathbf{F}^*$ are square matrices of dimension $N$, and the solution of system (7.9) is identical with that of the system of the BACM

$$\mathbf{F}\mathbf{b}^{(N)} = \mathbf{h}$$

Note that the elements of the matrix $\mathbf{F}$ and the vector $\mathbf{h}$ depend on $N$.

All the results of Subsection 7.1 as to when the approximate solution computed by the BACM converges to the exact solution, as well as the stability of the BACM, remain valid for the BACM constructed here.

## 8. CONCLUSION

The approach proposed above is general and suitable for a large range of problems (see [10, 11]). We have developed stable and convergent BAMLS and BACM to approximate solutions of Dirichlet boundary-value problems for harmonic and biharmonic equations, for the case in which the smoothness of the boundary and of the functions given on the boundary do not enable us to formulate a weak boundary-value problem [12, 13]. When constructing the BAMLS and BACM for such problems, use is made of the so-called "very weak formulation of the problem," or of the formulation of the problem in weight spaces of the functions given on the boundary [1, 2].

Compared with existing methods for approximating solutions to many problems, this approach enables us to reduce the Euclidean dimension of the problem by 1, due to the transition to the boundary of the domain; this considerably facilitates the algorithmization of the method. Unlike the finite-element method and the grid method, there is no need for discretization of the domain, which also facilitates the algorithmization process and is an advantage in solving problems with variable boundaries (optimal projection problems). Unlike the boundary-element method, this method yields a solution for the entire domain, without additional computations. Its computational stability yields an approximate solution of satisfactory accuracy.

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